Orbital Functional Geometry XII

Extension to Complex Functions: Winding Number, Topological Sectors, and Gauge Structure

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Abstract

We extend the Orbital Space $\mathcal{O}$ from positive real functions to complex-valued functions $f : S^1 \to \mathbb{C}^*$.

For $f \in \mathbb{C}^*$, write $u = \ln f = \ln|f| + i \arg f = \phi + i\alpha$. The orbital Schrödinger equation, linear in u, decomposes into a coupled system on the amplitude $\phi = \ln|f|$ and phase $\alpha = \arg f$:

$$\dot{\alpha} = \frac{\hbar}{2m}\phi'', \qquad \dot{\phi} = -\frac{\hbar}{2m}\alpha''$$

The linearisation $u = \ln f$ holds in the complex setting without modification.

The stationary states are $f_n(\theta) = e^{e^{in\theta}}$, $n \in \mathbb{Z}$: functions whose modulus is a von Mises distribution and whose phase is sinusoidal. The energy spectrum remains $E_n = \hbar^2 n^2/2m$, but the infinite degeneracy of the real case reduces to degeneracy 2 (states $\pm n$), exactly as for the rigid rotor.

The multivaluedness of $\arg f$ introduces the winding number $w(f) = \frac{1}{2\pi}\oint_{S^1} d(\arg f) \in \mathbb{Z}$ as a new topological invariant. The complex orbital space decomposes into topological sectors:

$$\mathcal{O}_{\mathbb{C}} = \bigsqcup_{k\in\mathbb{Z}} \mathcal{O}_k, \qquad \mathcal{O}_k = \{f : S^1 \to \mathbb{C}^* : w(f) = k\}$$

Each sector $\mathcal{O}_k$ is homeomorphic to $\mathcal{O}_0$ via $f \mapsto f \cdot e^{-ik\theta}$. The complex orbital space carries a natural $U(1)$ gauge structure on S^1, with transitions between sectors corresponding to topological instantons of charge 1.

Contents

1 The Complex Extension **4**

2 Orbital Dynamics in the Complex Case **4**

3 Stationary States and Spectrum **4**

4 The Winding Number **5**

5 Topological Sectors **5**

6 Gauge Structure **6**

Introduction

The real Orbital Space $\mathcal{O}$ of Papers I–XI requires $f > 0$ everywhere, so that $\ln f$ is real-valued. This restriction is natural for the original construction but excludes complex-valued functions, which are central to quantum mechanics and to the theory of analytic functions on S^1.

This paper extends $\mathcal{O}$ to $f : S^1 \to \mathbb{C}^*$. The extension is guided by a single principle: preserve the linearisation $u = \ln f$. This forces the complex logarithm and introduces the winding number as an unavoidable topological invariant.

The result is a $\mathbb{Z}$-graded space with a $U(1)$ gauge structure — a richer object than $\mathcal{O}$, but built on the same foundation.

1 The Complex Extension

Definition 1 (Complex orbital space)**.** *The* complex orbital space *is:*

$$\mathcal{O}_{\mathbb{C}} = \{f : S^1 \to \mathbb{C}^* \text{ analytic}\}$$

For $f \in \mathcal{O}_{\mathbb{C}}$, the complex logarithm is $u = \ln f = \phi + i\alpha$ where $\phi = \ln|f|$ and $\alpha = \arg f$.

Remark 1 (Multivaluedness)**.** *The function $\alpha = \arg f$ is defined modulo 2π. On a full circuit of S^1, it may acquire a shift of $2\pi k$ for some $k \in \mathbb{Z}$. This integer k is the winding number $w(f)$ and cannot be removed by a continuous deformation of f within $\mathcal{O}_{\mathbb{C}}$.*

2 Orbital Dynamics in the Complex Case

Theorem 1 (Linearisation in $\mathcal{O}_{\mathbb{C}}$)**.** *The substitution $u = \ln f = \phi + i\alpha$ transforms the orbital Schrödinger equation into the free equation on u:*

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial \theta^2}$$

The linearisation of Paper XI holds without modification in the complex setting.

Corollary 1 (Coupled amplitude-phase system)**.** *Separating real and imaginary parts of the free equation on $u = \phi + i\alpha$:*

$$\dot{\alpha} = \frac{\hbar}{2m}\phi'', \qquad \dot{\phi} = -\frac{\hbar}{2m}\alpha''$$

The amplitude $\phi = \ln|f|$ is driven by the curvature of the phase; the phase $\alpha = \arg f$ is driven by the curvature of the amplitude. The two are inseparable in the complex extension.

Remark 2 (Amplitude-phase coupling)**.** *In the real case ($\alpha = 0$), the two equations decouple and reduce to $\dot{\phi} = 0$ (stationary) or the free equation on ϕ alone. The complex extension introduces a non-trivial coupling: a spatially varying phase induces amplitude dynamics, and vice versa. This coupling is linear in u and has no analogue in standard linear quantum mechanics, where amplitude and phase decouple via $\psi = |\psi|e^{iS/\hbar}$ only in the WKB limit.*

3 Stationary States and Spectrum

Theorem 2 (Complex eigenfunctions)**.** *The stationary states in $\mathcal{O}_{\mathbb{C}}$ with winding number $w = 0$ are:*

$$f_n(\theta) = e^{e^{in\theta}}, \quad n \in \mathbb{Z}$$

Explicitly: $|f_n(\theta)| = e^{\cos(n\theta)}$ (von Mises modulus) and $\arg f_n(\theta) = \sin(n\theta)$ (sinusoidal phase).

Proof. Setting $u(\theta, t) = v(\theta)e^{-i\omega t}$ in the free equation gives $v'' = (2im\omega/\hbar)v$. For $\omega = -\hbar n^2/2m$ (negative, giving oscillatory solutions): $v'' = -n^2 v$, so $v(\theta) = e^{in\theta}$ (periodic on S^1). Then $f_n = e^{v_n} = e^{e^{in\theta}}$. □

Theorem 3 (Spectrum and degeneracy reduction). *The energy spectrum in $\mathcal{O}_{\mathbb{C}}$ is $E_n = \hbar^2 n^2/2m$, $n \in \mathbb{Z}$.*

For each $|n| \geq 1$, the degeneracy is exactly 2: the states f_n and f_{-n} have the same energy and are distinct. The infinite degeneracy of the real case (parametrised by $(R, \phi) \in \mathbb{R}^+ \times [0, 2\pi)$) reduces to degeneracy 2 in the complex extension.

Remark 3 (Mechanism of degeneracy reduction). *In the real case, the parameter $R > 0$ was free: $e^{R\cos(n\theta-\phi)}$ was an eigenfunction for any R. In the complex case, the periodicity condition on $u = e^{in\theta}$ fixes the amplitude to $|u| = 1$, removing the freedom in R. The extension to $\mathbb{C}^*$ selects a canonical representative at each energy level.*

$\lvert n\rvert$	E_n	Eigenfunctions	Degeneracy
0	0	$f = c \in \mathbb{C}^*$	1
1	$\hbar^2/2m$	$e^{e^{i\theta}},\ e^{e^{-i\theta}}$	2
2	$2\hbar^2/m$	$e^{e^{2i\theta}},\ e^{e^{-2i\theta}}$	2
n	$\hbar^2 n^2/2m$	$e^{e^{in\theta}},\ e^{e^{-in\theta}}$	2

4 The Winding Number

Definition 2 (Winding number). *For $f \in \mathcal{O}_{\mathbb{C}}$, the* winding number *is:*

$$w(f) = \frac{1}{2\pi} \oint_{S^1} d(\arg f) = \frac{1}{2\pi i} \oint_{S^1} \frac{f'}{f}\, d\theta \in \mathbb{Z}$$

It counts the number of times f winds around $0 \in \mathbb{C}$ as θ traverses S^1.

Lemma 1 (Winding of eigenfunctions). *The eigenfunctions $f_n = e^{e^{in\theta}}$ have winding number $w(f_n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. $\arg f_n(\theta) = \sin(n\theta)$, which is periodic with no net drift: $\sin(n \cdot 2\pi) = \sin(0) = 0$. Hence $w(f_n) = 0$. □

Lemma 2 (Generating winding-k functions). *Functions of winding number $k \in \mathbb{Z}$ are generated by:*

$$f(\theta) = e^{e^{in\theta}+ik\theta}, \quad w(f) = k$$

The factor $e^{ik\theta}$ contributes exactly k to the winding number.

5 Topological Sectors

Definition 3 (Topological sectors). *For each $k \in \mathbb{Z}$, the* topological sector of charge k *is:*

$$\mathcal{O}_k = \{f \in \mathcal{O}_{\mathbb{C}} : w(f) = k\}$$

The complex orbital space decomposes as:

$$\mathcal{O}_{\mathbb{C}} = \bigsqcup_{k\in\mathbb{Z}} \mathcal{O}_k$$

Theorem 4 ($\mathbb{Z}$-grading). *Each sector $\mathcal{O}_k$ is homeomorphic to $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}} \cap \{w = 0\}$ via the map:*

$$\varphi_k : \mathcal{O}_k \to \mathcal{O}_0, \quad f \mapsto f \cdot e^{-ik\theta}$$

The complex orbital space is $\mathbb{Z}$-graded: the sector structure is preserved under composition, $w(fg) = w(f) + w(g)$.

6 Gauge Structure

Theorem 5 ($U(1)$ gauge structure)**.** *The complex orbital space $\mathcal{O}_{\mathbb{C}}$ carries a natural $U(1)$ gauge structure on S^1:*

1. *The gauge group is $\mathcal{G} = \{g : S^1 \to U(1)\} \cong \mathrm{Map}(S^1, U(1))$*
2. *The gauge action $f \mapsto g \cdot f$ for $g \in \mathcal{G}$ preserves $|f|$ and shifts $\arg f$ by $\arg g$*
3. *The winding number $w(f)$ is a gauge-invariant topological charge: $w(g \cdot f) = w(g) + w(f)$*
4. *Transitions between sectors $\mathcal{O}_k \to \mathcal{O}_{k+1}$ correspond to gauge transformations $g(\theta) = e^{i\theta}$ of winding number 1 — topological instantons of minimal charge*

Remark 4 (Instantons in $\mathcal{O}_{\mathbb{C}}$)**.** *A topological instanton of charge 1 is a path in $\mathcal{O}_{\mathbb{C}}$ connecting $\mathcal{O}_k$ to $\mathcal{O}_{k+1}$ that cannot be continuously deformed to a constant path. The minimal instanton is $g_t(\theta) = e^{it\theta}$ for $t \in [0, 1]$, which interpolates between $g_0 = 1$ (sector $\mathcal{O}_k$) and $g_1 = e^{i\theta}$ (shift to $\mathcal{O}_{k+1}$). These instantons are the topological excitations of the complex orbital space.*

Remark 5 (Real case as $w = 0$ sector)**.** *The real Orbital Space $\mathcal{O}$ of Papers I–XI is the sector $\mathcal{O}_0$ restricted to $f > 0$ (so that $\alpha = \arg f = 0$). It is the topologically trivial sector of $\mathcal{O}_{\mathbb{C}}$. The entire structure of Papers I–XI lives in $\mathcal{O}_0$; the complex extension reveals the $\mathbb{Z}$-grading that was invisible in the real case.*

Summary of New Results

Result	Statement	Status
Complex extension	$\mathcal{O}_{\mathbb{C}} = \{f : S^1 \to \mathbb{C}^*\}$	Established
Linearisation holds	$u = \ln f$ gives free equation on $\mathbb{C}$	Established
Coupled system	$\dot{\alpha} = (\hbar/2m)\phi''$, $\dot{\phi} = -(\hbar/2m)\alpha''$	Established
Complex eigenfunctions	$f_n = e^{e^{in\theta}}$, modulus von Mises, phase sinusoidal	Established
Spectrum unchanged	$E_n = \hbar^2 n^2/2m$, $n \in \mathbb{Z}$	Established
Degeneracy reduction	$\infty \to 2$ by complex extension	Established
Winding number	$w(f) = \frac{1}{2\pi i} \oint f'/f \, d\theta \in \mathbb{Z}$	Established
Eigenfunctions at $w = 0$	$w(f_n) = 0$ for all n	Established
Topological sectors	$\mathcal{O}_{\mathbb{C}} = \bigsqcup_k \mathcal{O}_k$	Established
$\mathbb{Z}$-grading	$\mathcal{O}_k \cong \mathcal{O}_0$ via $f \mapsto fe^{-ik\theta}$	Established
$U(1)$ gauge structure	Gauge group, charge, instantons	Established
Real case as $\mathcal{O}_0$	Papers I–XI live in topologically trivial sector	Established
Spectral theory of $\mathcal{O}_k$	Eigenfunctions with $w = k$	Open
Instanton calculus	Amplitudes for sector transitions	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–XI (2026). The winding number emerged as an unavoidable consequence of extending $\ln f$ to $\mathbb{C}^*$: it

cannot be removed by any continuous deformation. The $\mathbb{Z}$-grading and gauge structure followed from the topology of $\mathrm{Map}(S^1, \mathbb{C}^*) \simeq \mathbb{Z}$. These are not imposed structures — they are what the extension forces.

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References

[1] J. Brindel, *Orbital Functional Geometry I–XI*, Preprints, Cotonou, 2026.

[2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators*, Ann. Math. 87 (1968), 484–530.

[3] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, 1986.

[4] D. Bohm, *A suggested interpretation of the quantum theory in terms of hidden variables*, Phys. Rev. 85 (1952), 166–179.

[5] M. V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. R. Soc. London A 392 (1984), 45–57.